Probability Theory

An Introduction

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Outline

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Set Theory

Foundation of all mathematics

In the late XIX century and early XX century, the problem of securing the foundation of all mathematics arose. It resulted in, among other great works, proposing Set Theory as the foundational system for all mathematics. Now it is an accepted result. With formal axioms and equivalent theories is ZFC, from which we build and assume that everything in math is a set.

As such, even counting numbers $\mathbb{N}$ are considered sets, by letting:

- $0 := \emptyset$
- $S(n) := n \cup \{n\}$, here S is called the successor function.

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Functions

Definition

Given two sets X and Y, and $f \subseteq X \times Y$, f is called a function with domain X and codomain Y if and only if:

- $(\forall x \in X, \exists y \in Y)((x, y) \in f)$
- $(\forall (x_1, y_1), (x_2, y_2) \in f)(x_1 = x_2 \to y_1 = y_2)$

For such, we write $f : X \longrightarrow Y$. Given a pair $(x, y) \in f$ we say $f(x) := y$.

Functions

Surjection and Injection

Given $f : X \longrightarrow Y$:

- we say that f is surjective if and only if (iff): $(\forall y \in Y, \exists x \in X)(f(x) = y)$
- we say that f is injective iff: $(\forall (x_1, y_1), (x_2, y_2) \in f)(y_1 = y_2 \to x_1 = x_2)$

If f is surjective and injective, it is called bijective.

Functions

Image and Preimage

Given $f : X \longrightarrow Y$ and $B \subseteq Y$:

- we define the image of f as $\mathrm{Im}(f) := \{y \in Y \mid (\exists x \in X)(f(x) = y)\}$
- we define the preimage of B under f as $\Pi_f(B) := \{x \in X \mid f(x) \in B\}$

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Indices

Definition

If $I \subseteq \mathbb{N}$ and $f : I \longrightarrow A$ is bijective, we say that set A is indexed by f and is countable; for shorthand, we denote this fact as $\{A_i\}$, and for convenience $f(i) = a \to a_i := a$.

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Set Exponentiation

Definition

Given two sets X and Y, the set exponentiation of X to the Y is defined as $X^Y := \{f \mid f : Y \longrightarrow X\}$.

Power Set

Definition

Given a set A, we define its power set $\mathcal{P}(A) := \{p \mid p \subseteq A\}$; equivalently, and hence its notation elsewhere: $\mathcal{P}(A) := \{x \mid (\exists s \in 2^A)(s(a) = 1 \rightarrow a \in x)\}$.

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σ-Algebras

Definition

If A is a set, $\Sigma \subseteq \mathcal{P}(A)$ is called a σ-algebra over A iff:

- $A \in \Sigma$
- $(\forall \sigma \in \Sigma)(A \setminus \sigma \in \Sigma)$
- $$(\forall \{\Xi_i\} \subseteq \Sigma)\left(\bigcup_{i \in I} \xi_i \in \Sigma\right)$$

The duple (A, Σ) is called a measurable space.

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Measurable Functions

Definition

Given two measurable spaces (A, Σ), (B, T) a function $f : A \longrightarrow B$ is called a measurable function iff $(\forall \tau \in T)(\Pi_f(\tau) \in \Sigma)$.

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Measures

Definition

Given a measurable space (A, Σ), $\mu : \Sigma \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ is called a measure iff:

- $\mu(\emptyset) = 0$
- $(\forall \sigma \in \Sigma)(\mu(\sigma) \geq 0)$
- $(\forall \{\Xi_i\} \subseteq \Sigma)((\forall i, j \in I)(i \neq j \rightarrow \xi_i \cap \xi_j = \emptyset) \rightarrow \mu(\bigcup_{i \in I} \xi_i) = \sum_{i \in I} \mu(\xi_i))$

The triple (A, Σ, μ) is called a measure space.

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Pushforward Measure

Definition

Given a measure space (A, Σ, μ), a measurable space (B, T) and a measurable function $f : A \longrightarrow B$, the pushforward measure of μ under f is defined as $f\#\mu : T \longrightarrow \mathrm{Im}(\mu)$ with correspondence $(f\#\mu)(\tau) := \mu(\Pi_f(\tau))$.

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Probability Measure

Definition

Given a measure space $(\Omega, \mathcal{F}, P)$, P is called a probability measure iff $P(A) = 1$. In this case the measure space is also called a probability space and Ω and $\mathcal{F}$ are called the sample space and the event set, respectively.

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Random Variable

Definition

Given a probability space $(\Omega, \mathcal{F}, P)$ and a measure space (A, Σ), any measurable function $X : \Omega \longrightarrow A$ is called a random variable.

The pushforward measure of P under X is called the probability distribution of X, namely: $X\#P$.

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